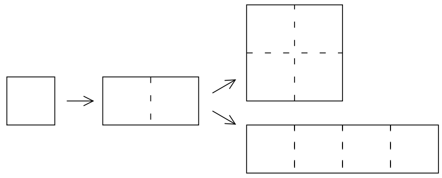
UK Junior Mathematical Olympiad 2008 Solutions

A1 3 A two-digit prime cannot end with a 2, 5 or 8. So we need only check 23, 53 and 83, each of which is prime.

A2 30 cm As the square has a perimeter of 12 cm, it must have a side length of 3cm. Unfolding the square once gives a 6 cm × 3 cm rectangle.

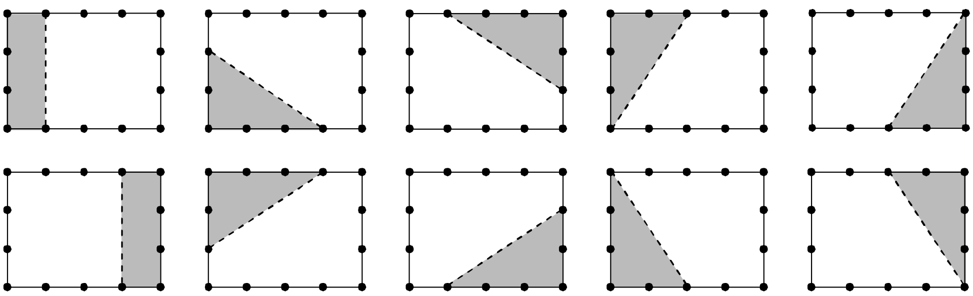
This rectangle can then be unfolded about either a long edge or a short edge resulting in a 6 cm × 6 cm square or a 12 cm × 3 cm rectangle. The perimeters of these are 24 cm and 30 cm respectively.



A3 $x = 36$ Rearranging the equation, we have $\frac{1}{x} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{12} - \frac{1}{18} = \frac{36 - 18 - 12 - 3 - 2}{36} = \frac{1}{36}$, and so $x = 36$.

A4 9 The digits of a three-digit number whose product is 6 must be either two 1s and a 6, or a 1, 2 and 3. In the first case, there are three choices where to put the 6; in the other case, there are three choices for the first digit and then two choices for the second. Hence, altogether, there are 9 such numbers: 116, 161, 611 and 123, 132, 213, 231, 312, 321.

A5 10 The smaller area must be either a rectangle or a triangle of area 3. There are only two such rectangles, one at each end, as shown. The triangle needs to have base 2 and height 3 (or vice versa). There are two positions on each edge for the base. So we get eight triangles this way, as shown.



A6 8 Factorising 1600 into the product of its prime factors gives $1600 = 2^6 \times 5^2$. The factors of 1600 are either 1 or the numbers less than or equal to 1600 whose prime factors are only 2 or 5. Of the latter, the square numbers are those where the powers of 2 and of 5 are even. So the square numbers that are factors of 1600 are 1, 2^2, 2^4, 2^6, 5^2, $2^2 \times 5^2$, $2^4 \times 5^2$ and 1600 itself

[*Alternatively*: A factor of 1600 is a square number if and only if its square root is a factor of $\sqrt{1600} = 40$. There are eight factors of 40: 1, 2, 4, 5, 8, 10, 20 and 40.]

A7 $\boldsymbol{x = 0}$ As the bottom row and the diagonal running from 6 to x have the same sum, $x + 9$, the middle square must contain the number 3. Thus the remaining numbers in the top row must be $x + 2$ (from the second column) and $x + 1$ (from the diagonal). Considering the top row, we have $x + 1 + x + 2 + 6 = x + 9$, thus $x = 0$.

$x + 1$	$x + 2$	**6**
	3	
$\boldsymbol{x}$	**4**	**5**

[*Alternatively*: It is well known that the magic sum of a 3×3 magic square is three times the middle number. Once the middle number is known it follows that $x = 0$.]

A8 1 After Annie has taken her last 8 sweets, let x be the number of sweets that Clarrie, Lizzie and Annie have and s be the number of sweets left for Danni, where $0 < s < 8$. After Danni takes the remaining s sweets, she has a total of $x - 8 + s$.

Let the three girls each give e sweets to Danni, so that

$$x - e = x - 8 + s + 3e$$

which gives

$$4e = 8 - s.$$

Because $0 < 4e < 8$, we have $e = 1$.

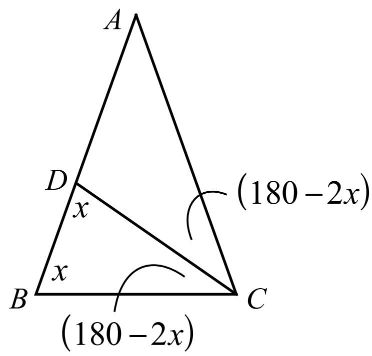
[*Alternatively*: Since the girls end up with equal numbers of sweets, the total number is a multiple of 4. They have all picked multiples of 8 except for the final number picked by Danni, and so she gets 4 sweets in her last turn. If each of the three other girls gives Danni 1 sweet, they all effectively gain 7 sweets in their last turn, whereas if they each gave more than 1 sweet, Danni would have more than them.]

A9 108° Triangle BCD is isosceles, so let $\angle DBC = \angle BDC = x°$.

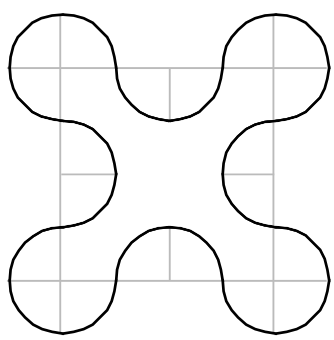
Then $\angle BCD = \angle ACD = (180 - 2x)°$.

Since triangle ABC is also isosceles, $\angle ACB = \angle ABC = x°$, and so $x = 180 - 2x + 180 - 2x$, and hence $x = 72$.

Thus $\angle CDA = (180 - x)° = 108°$.



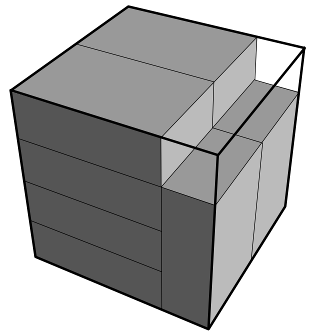
A10 $\boldsymbol{(64 + 4\pi)}$ cm^2 Adding the construction lines shown, we can see that the area of the shape
= the area of the square + area of 4 quarter-circles
$= (4 \times 2)^2 + \pi \times 2^2 = (64 + 4\pi)\,\text{cm}^2$.



B1 Tamsin has a selection of cubical boxes whose internal dimensions are whole numbers of centimetres, that is, 1 cm × 1 cm × 1 cm, 2 cm × 2 cm × 2 cm, and so on.

What are the dimensions of the smallest of these boxes in which Tamsin could fit ten rectangular blocks each measuring 3 cm × 2 cm × 1 cm without the blocks extending outside the box?

Solution

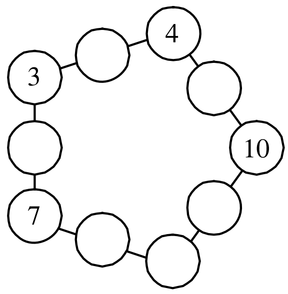


The total volume of 10 blocks, each 3 cm × 2 cm × 1 cm, is $10 \times 3 \times 2 \times 1 = 60$ cm^3.
The volumes of a 1 cm cube, a 2 cm cube and a 3 cm cube are 1 cm^3, 8 cm^3 and 27 cm^3 respectively so these cubes cannot contain the 10 blocks.

So the smallest *possible* cube is a 4 cm cube and the diagram on the right shows how such a cube can hold the 10 blocks.

B2 Each of the numbers from 1 to 10 is to be placed in the circles so that the sum of each line of three numbers is equal to T. Four numbers have already been entered.

Find all the possible values of T.



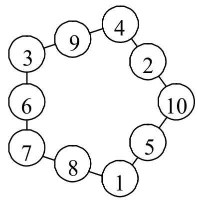
Solution

Let x be the number in the circle in the unfilled bottom corner of the pentagon.

All ten numbers are used once and only once and so the sum of the numbers in all ten circles is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$.

Now, consider the numbers as they appear in the five lines of three circles. When these five sets of numbers are added, the corner numbers (3, 4, 7, 10, x) will be included twice so the sum is

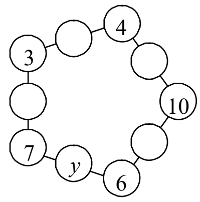
$$5T - 3 - 4 - 7 - 10 - x = 5T - x - 24.$$



Equating the two sums, we have $5T - x - 24 = 55$ which gives $5T = 79 + x$.

Since T is an integer, $5T$ is a multiple of 5, which means that x must be either 1 or 6.

If $x = 1$ then $T = 16$, while if $x = 6$, $T = 17$.



The first diagram shows that $x = 1$, $T = 16$ is a possibility, while in the second diagram, $x = 6$, $T = 17$ forces $y = 4$, repeating the 4 in the top corner.

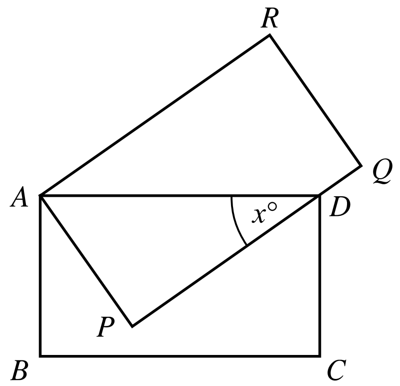
Thus the only possible value of T is 16.

[*Alternatively*: Let the number in the circle between 4 and 10 be a. Then the total for each line T is $a + 14$, and so the number in the circle between 3 and 4 is $a + 7$. Yet this is between 1 and 10 and cannot be 7 or 10, and so $a = 1$ or $a = 2$.

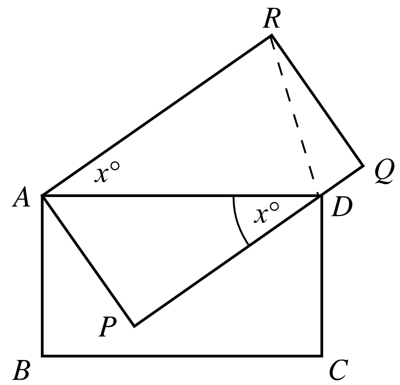
If we try $a = 1$, we get $T = 15$, but there is no way of completing the other line including the 10, without reusing either the 3 or the 4.

If we try $a = 2$, we obtain the solution shown above, and hence $T = 16$ is the only possibility.]

B3 In the diagram $ABCD$ and $APQR$ are congruent rectangles. The side PQ passes through the point D and $\angle PDA = x°$.
Find an expression for $\angle DRQ$ in terms of x.



Solution
Since $APQR$ is a rectangle, the lines AR and PQ are parallel.
Thus $\angle PDA = \angle DAR$ (alternate angles) so $\angle DAR = x°$.



Since $DA = RA$, triangle DAR is isosceles and so
$\angle ARD = \angle ADR = \frac{1}{2}(180 - x)° = \left(90 - \frac{1}{2}x\right)°$.

Hence, since $\angle ARQ = 90°$, $\angle DRQ = \frac{1}{2}x°$.

B4 For each positive two-digit number, Jack subtracts the units digit from the tens digit; for example, the number 34 gives $3 - 4 = -1$.
What is the sum of all his results?

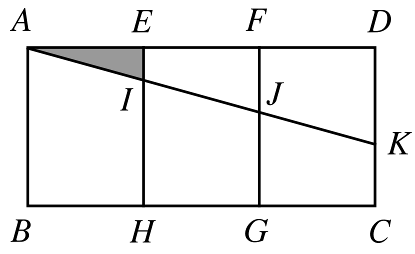
Solution
The two-digit numbers run from 10 to 99.
We may categorise these numbers into the following four sets:
$P = \{$the numbers ab where a and b are strictly positive and $a < b\}$
$Q = \{$the numbers ab where a and b are strictly positive and $b < a\}$
$R = \{$the palindromes $aa\}$
$S = \{$the numbers ab where $a > 0$ and $b = 0\}$.
For numbers 'ab', the units digit subtracted from the tens digit is $a - b$.

The result $a - b$ from each of the numbers in set P can be matched with the result $b - a$ from each corresponding number in the set Q, giving a total of $(a - b) + (b - a) = 0$.
For each of the numbers in set R the result is $a - a$ which is 0.
Finally, when we consider all of the numbers in set S, the result is
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$. So the total is 45.

[*Alternatively*: The units digits for all the two-digit integers comprise the numbers 0, 1, ..., 9 each 9 times (coming after 1, 2, ..., 9).
The tens digits are 1, 2, ..., 8, 9, each coming 10 times (before 0, 1, ..., 9).
The difference is thus $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.]

B5 In the diagram, the rectangle $ABCD$ is divided into three congruent rectangles. The line segment JK divides $CDFG$ into two parts of equal area.

What is the area of triangle AEI as a fraction of the area of $ABCD$?



Solution

Let a = the area of AEI.

We have $AF = 2 \times AE$ and $AD = 3 \times AE$. Also the triangles AEI, AFJ and ADK are similar since $\angle A$ is common and the angles at E, F and D are right-angles. Hence the area of $ADK = 9a$, the area of $AFJ = 4a$ and so the area of $DFJK = 9a - 4a = 5a$.

So the area of $DFGC = 2 \times 5a = 10a$ and the area of $ABCD = 3 \times 10a = 30a$.

Thus $\dfrac{\text{area } AEI}{\text{area } ABCD} = \dfrac{1}{30}$.

B6 In a sequence of positive integers, each term is larger than the previous term. Also, after the first two terms, each term is the sum of the previous two terms.

The eighth term of the sequence is 390. What is the ninth term?

Solution

Let the first two terms be the positive integers a and b. So the first nine terms of the sequence are:

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b,\ 5a + 8b,\ 8a + 13b,\ 13a + 21b.$$

Thus $\quad 8a + 13b = 390$

giving $\quad 8a = 390 - 13b = 13(30 - b). \qquad (*)$

So $8a$ is a multiple of 13, and hence a itself is a multiple of 13.

Since $8a = 390 - 13b$, if $a \geqslant 26$ then $390 - 13b \geqslant 8 \times 26$, i.e. $13b \leqslant 390 - 208 = 182$ which means $b \leqslant 14 < a$ and the sequence would not be increasing.

So $a = 13$ and $13b = 390 - 13 \times 8$ giving $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.

[*Alternative ending*: From the starred equation, $30 - b$ is a multiple of 8, and hence could be 0, 8, 16 or 24. This means that b could be 30, 22, 14 or 6, from which the corresponding values of a are 0, 13, 26 and 39.

We know that $0 < a < b$, so the only possibility here is $a = 13$ and $b = 22$.

Thus the ninth term is $13a + 21b = 13 \times 13 + 21 \times 22 = 169 + 462 = 631$.]